

(56) 複數

請看以下的方程式：

$$x^2 + 3x + 1 = 0$$

我們可以用公式來求答案：

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 1}}{2} = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2} = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}i$$

因此，我們得到了一個 $-\frac{3}{2} \pm \frac{\sqrt{5}}{2}i$ 的數。這個數比純的虛數要複雜一點，因為它有實數部分。

如果 a 與 b 都是實數，則 $a + bi$ 是一個複數。因此，以下的數都是複數：

$$1 + 2i$$

$$-3 + 5i$$

$$-\frac{1}{2} - \frac{2}{3}i$$

$$5 + \sqrt{6}i$$

$$\sqrt{7} - \sqrt{6}i$$

複數的加減乘除

(1)

$$(a) (3 + 5i) + (4 - 6i) = (3 + 4) + (5i - 6i) = 7 + (-i) = 7 - i$$

$$(b) -(2 + 3i) + (5 - 7i) = -2 - 3i + 5 - 7i = (-2 + 5) + (-3i - 7i) = 3 + (-10i) = 3 - 10i$$

$$(c) (5 - 3i) - (-7 + 9i) = 5 - (-7) - 3i - 9i = 12 - 12i$$

$$(d) (8 + 3i) - (\sqrt{2} - \sqrt{3}i) = (8 - \sqrt{2}) + (3i + \sqrt{3}i) = (8 - \sqrt{2}) + (3 + \sqrt{3})i$$

(2)

$$(a) (3 + 5i)(4 - 6i) = (3 \times 4) + 3(-6i) + (5i) \times 4 + (5i) \times (-6i) = 12 - 18i +$$

$$20i - 30i^2 = 12 + 2i - 30(-1) = 42 + 2i$$

$$(b) (2 + 3i)(5 - 7i) = (2 \times 5) + 2(-7i) + (3i)5 + (3i)(-7i) = 10 - 14i + 15i - 21i^2 = 10 + i - 21(-1) = 31 + i$$

$$(c) (3 + i)(3 - i) = 9 - i^2 = 9 + 1 = 10$$

$$(d) (3i + 1)(3i - 1) = 9i^2 - 1 = -9 - 1 = -10$$

$$(e) (i + 1)^2 = i^2 + 2i + 1 = -1 + 2i + 1 = 2i$$

$$(f) (3i - 2)^2 = 9i^2 - 12i + 4 = -9 - 12i + 4 = -5 - 12i$$

$$(g) (-i + 2)^2 = (-i)^2 - 4i + 4 = i^2 - 4i + 4 = 3 - 4i$$

$$(h) (2 + \sqrt{2}i)(\sqrt{8}i + 3) = 2\sqrt{8}i + 6 + \sqrt{2}\sqrt{8}i^2 + 3\sqrt{2}i = 6 + \sqrt{2}\sqrt{8}(-1) + 2\sqrt{8}i + 3\sqrt{2}i = (6 - \sqrt{16}) + 2 \times 2\sqrt{2}i + 3\sqrt{2}i = (6 - 4) + 7\sqrt{2}i = 2 + 7\sqrt{2}i$$

(3) 除法

$$(a) \frac{3-i}{1-i} = \frac{(3-i)(1+i)}{(1-i)(1+i)} = \frac{3+3i-i-i^2}{1-i^2} = \frac{3-(-1)+2i}{1-(-1)} = \frac{4+2i}{2} = 2 + i$$

$$(b) \frac{1}{1-i} = \frac{(1+i)}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1+i}{1-(-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{i}{2}$$

$$(c) \frac{4+i}{3-i} = \frac{(4+i)(3+i)}{(3-i)(3+i)} = \frac{12+4i+3i+i^2}{9-i^2} = \frac{12-1+7i}{9-(-1)} = \frac{11+7i}{10} = \frac{11}{10} + \frac{7}{10}i$$

$$(d) \frac{1}{(3+i)^2} = \frac{1}{9+6i+i^2} = \frac{1}{9-1+6i} = \frac{1}{8+6i} = \frac{8-6i}{(8+6i)(8-6i)} = \frac{8-6i}{64-(6i)^2} = \frac{8-6i}{64+36} = \frac{8-6i}{100} = \frac{8}{100} -$$

$$\frac{6}{100}i = \frac{2}{25} - \frac{3}{50}i$$

$$(e) \frac{i+1}{3i-5} = \frac{(i+1)(3i+5)}{(3i-5)(3i+5)} = \frac{3i^2+5i+3i+5}{(3i)^2-25} = \frac{-3+5+8i}{-9-25} = \frac{2+8i}{-34} = -\frac{1}{17} - \frac{4}{17}i$$

$$(f) \frac{1}{(i+2)^2} = \frac{1}{i^2+4i+4} = \frac{1}{-1+4i+4} = \frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3-4i}{9-(4i)^2} = \frac{3-4i}{9+16} = \frac{3-4i}{25} = \frac{3}{25} -$$

$$\frac{4}{25}i$$

$$(g) \frac{2+i}{3-5i} = \frac{(2+i)(3+5i)}{(3-5i)(3+5i)} = \frac{6+10i+3i+5i^2}{9-(5i)^2} = \frac{6-5+13i}{9+25} = \frac{1+13i}{34} = \frac{1}{34} + \frac{13}{34}i$$

(4)一元二次方程式

$$(a)x^2 + x + 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$$

我們可以以 $x = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ 查看答案是否正確

$$x^2 + x + 1$$

$$= \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right)^2 + \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) + 1$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2 - \frac{1}{2} + \frac{\sqrt{3}}{2}i + 1$$

$$= \frac{1}{4} - \frac{3}{4} - \frac{1}{2} + 1 = 0$$

可見得 $x = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$ 是正確的

$$(b) x^2 - 2x + 2 = 0$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

我們以 $x = 1 - i$ 來看答案是否正確

$$x^2 - 2x + 2$$

$$= (1 - i)^2 - 2(1 - i) + 2$$

$$= 1 - 2i + i^2 - 2 + 2i + 2$$

$$= 1 + i^2 = 0$$

$$(c) \ x^2 - x + 2 = 0$$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 2}}{2} = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

我們以 $x = \frac{1}{2} + \frac{\sqrt{7}}{2}i$ 來看答案是否正確

$$x^2 - x + 2$$

$$\begin{aligned} &= \left(\frac{1}{2} + \frac{\sqrt{7}}{2}i\right)^2 - \left(\frac{1}{2} + \frac{\sqrt{7}}{2}i\right) + 2 \\ &= \frac{1}{4} + \frac{\sqrt{7}}{2}i + \frac{7}{4}i^2 - \frac{1}{2} - \frac{\sqrt{7}}{2}i + 2 \\ &= \frac{1}{4} - \frac{7}{4} - \frac{1}{2} + 2 = 0 \end{aligned}$$

$$(d) \ 2x^2 - x + 1 = 0$$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 2 \times 1}}{2 \times 2} = \frac{1 \pm \sqrt{-7}}{4} = \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$$

我們以 $x = \frac{1}{4} + \frac{\sqrt{7}}{4}i$ 來看答案是否正確

$$2x^2 - x + 1$$

$$\begin{aligned} &= 2\left(\frac{1}{4} + \frac{\sqrt{7}}{4}i\right)^2 - \left(\frac{1}{4} + \frac{\sqrt{7}}{4}i\right) + 1 \\ &= 2\left(\frac{1}{16} + \frac{\sqrt{7}}{8}i + \frac{7}{16}i^2\right) - \frac{1}{4} - \frac{\sqrt{7}}{4}i + 1 \\ &= \frac{1}{8} + \frac{\sqrt{7}}{4}i - \frac{7}{8} - \frac{1}{4} - \frac{\sqrt{7}}{4}i + 1 \\ &= \frac{1}{8} - \frac{7}{8} - \frac{1}{4} + 1 = 0 \end{aligned}$$

(5)帶虛數的方程式

$$(a) 2Z + 3i = 5$$

$$2Z = 5 - 3i$$

$$Z = \frac{5}{2} - \frac{3}{2}i$$

$$(b) 2Z - iZ = 5$$

$$Z(2 - i) = 5$$

$$Z = \frac{5}{2-i} = \frac{5(2+i)}{(2-i)(2+i)} = \frac{5(2+i)}{4-i^2} = \frac{5(2+i)}{5} = 2 + i$$

我們可以將 $Z = 2 + i$ 代入原方程式，看看答案對不對

$$2Z - iZ = 5$$

$$\text{左式} = 2(2 + i) - i(2 + i)$$

$$= 4 + 2i - 2i - i^2$$

$$= 4 - (-1) = 5 = \text{右式}$$

我們可知答案是對的

$$(c) (3 + i)Z = 5iZ - 3$$

$$(3 + i - 5i)Z = -3$$

$$(3 - 4i)Z = -3$$

$$Z = \frac{-3}{3-4i} = \frac{-3(3+4i)}{(3-4i)(3+4i)} = \frac{-9-12i}{9-(-16)} = \frac{-9}{25} - \frac{12}{25}i$$

將此答案代入原方程式

$$\text{左式} = (3 + i) \left(\frac{-9}{25} - \frac{12}{25}i \right)$$

$$= \frac{-27}{25} - \frac{36}{25}i - \frac{9}{25}i - \frac{12}{25}i^2$$

$$= \frac{-27}{25} + \frac{12}{25} - \frac{45}{25}i$$

$$= \frac{-15}{25} - \frac{45}{25}i = \frac{-3}{5} - \frac{9}{5}i$$

$$\text{右式} = 5i \left(\frac{-9}{25} - \frac{12}{25}i \right) - 3$$

$$= \frac{-9}{5}i - \frac{12}{5}i^2 - 3 = \frac{-9}{5}i - \frac{3}{5}$$

左式=右式，答案正確。

$$(d) (1+i)Z = 2 - Z$$

$$(1+i+1)Z = 2$$

$$(2+i)Z = 2$$

$$Z = \frac{2}{2+i} = \frac{2(2-i)}{(2+i)(2-i)} = \frac{4-2i}{4-(-1)} = \frac{4}{5} - \frac{2}{5}i$$

查看答案

$$\text{左式} = (1+i)Z$$

$$= (1+i)\left(\frac{4}{5} - \frac{2}{5}i\right)$$

$$= \frac{4}{5} - \frac{2}{5}i + \frac{4}{5}i - \frac{2}{5}i^2$$

$$= \frac{4}{5} + \frac{2}{5} + \frac{2}{5}i = \frac{6}{5} + \frac{2}{5}i$$

$$\text{右式} = 2 - Z$$

$$= 2 - \left(\frac{4}{5} - \frac{2}{5}i\right)$$

$$= 2 - \frac{4}{5} + \frac{2}{5}i = \frac{6}{5} + \frac{2}{5}i$$

左式=右式，答案正確。