

(29)有理數的指數定律

我們在過去常看到 $2^1 = 2, 2^2 = 4, 2^3 = 8$,我們不知道 $2^{\frac{1}{2}}, 2^{\frac{1}{3}}$ 或 $2^{\frac{1}{4}}$ 的意義，其實這並不困難，

$$\because \left(2^{\frac{1}{2}}\right)^2 = 2^{\frac{1}{2} \cdot 2} = 2^1 = 2$$

$$\therefore 2^{\frac{1}{2}} = \sqrt[2]{2}$$

同理

$$\left(2^{\frac{1}{3}}\right)^3 = 2^1 = 2$$

$$2^{\frac{1}{3}} = \sqrt[3]{2}$$

我們因此有下列的式子

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

1. $2^{\frac{1}{4}} = \sqrt[4]{2}$

2. $4^{\frac{1}{3}} = \sqrt[3]{4}$

3. $3^{\frac{1}{2}} = \sqrt{3}$

4. $5^{\frac{1}{5}} = \sqrt[5]{5}$

5. $100^{\frac{1}{6}} = \sqrt[6]{100}$

6. $36^{\frac{1}{2}} = \sqrt{36} = 3$

7. $8^{\frac{1}{3}} = \sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2} = 2$

8. $9^{\frac{1}{2}} = \sqrt{9} = \sqrt{3 \cdot 3} = 3$

9. $256^{\frac{1}{2}} = \sqrt{256} = \sqrt{16 \cdot 16} = 16$

10. $81^{\frac{1}{2}} = \sqrt{81} = \sqrt{9 \cdot 9} = 9$

11. $729^{\frac{1}{2}} = \sqrt{729} = \sqrt{27 \cdot 27} = 27$

$$12. 729^{\frac{1}{3}} = \sqrt[3]{729} = \sqrt[3]{9 * 9 * 9} = 9$$

$$13. 729^{\frac{1}{6}} = \sqrt[6]{729} = \sqrt[6]{3^6} = 3$$

$$14. 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$15. 125^{-\frac{1}{3}} = \frac{1}{125^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{125}} = \frac{1}{5}$$

$$16. 36^{-\frac{1}{2}} = \frac{1}{36^{\frac{1}{2}}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

$$17. 9^{-\frac{1}{2}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

$$18. 100^{\frac{1}{2}} = \sqrt{100} = 10$$

$$19. 100^{-\frac{1}{2}} = \frac{1}{100^{\frac{1}{2}}} = \frac{1}{\sqrt{100}} = \frac{1}{10}$$

$$20. 1000^{\frac{1}{3}} = \sqrt[3]{1000} = \sqrt[3]{10^3} = 10$$

我們現在看 $a^{\frac{m}{n}}$ ，有兩種方法解釋 $a^{\frac{m}{n}}$

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m \text{-----}(1)$$

$$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} \text{-----}(2)$$

假設 $a = 4, m = 3, n = 2$

$$a^{\frac{m}{n}} = 4^{\frac{3}{2}}$$

假設(1) $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = (\sqrt{4})^3 = 2^3 = 8$

假設(2) $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = \sqrt{4^3} = \sqrt{64} = 8$

可以看出(1)和(2)都是可以用的，比較常用的是(2)，我們先假設 $m > n$ ，在這種情況我們可以簡化原案。

我們要求 $a^{\frac{m}{n}}$

假設我們用公式(2)

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}}$$

$$\text{令 } m = n + (m - n)$$

$$a^{\frac{m}{n}} = (a^{n+(m-n)})^{\frac{1}{n}} = (a^n)^{\frac{1}{n}} * (a^{m-n})^{\frac{1}{n}} = a * \left(a^{\frac{1}{n}}\right)^{m-n}$$

因此可以得此下列公式 如果 $m > n$ 則

$$a^{\frac{m}{n}} = a * \left(a^{\frac{1}{n}}\right)^{m-n} \text{-----}(3)$$

$$1. 4^{\frac{3}{2}} = 4 * \left(4^{\frac{1}{2}}\right)^{3-2} = 4 * 2^1 = 4 * 2 = 8$$

$$2. 5^{\frac{4}{3}} = 5 * \left(5^{\frac{1}{3}}\right)^{4-3} = 5 * \left(\sqrt[3]{5}\right)^1 = 5 * \sqrt[3]{5} = 5\sqrt[3]{5}$$

$$3. 9^{\frac{3}{2}} = 9 * \left(9^{\frac{1}{2}}\right)^{3-2} = 9 * (3)^1 = 9 * 3 = 27$$

我們也可以直接計算 $9^{\frac{3}{2}} = \sqrt{9^3} = \sqrt{729} = 27$

$$4. 3^{\frac{3}{2}} = 3 * \left(3^{\frac{1}{2}}\right)^{3-2} = 3 * (\sqrt{3})^1 = 3\sqrt{3}$$

$$5. (a^2 + 2ab + b^2)^{\frac{3}{2}} = (a^2 + 2ab + b^2) \left((a^2 + 2ab + b^2)^{\frac{1}{2}}\right)^{3-2} = (a^2 + 2ab + b^2)(a + b)^1 = (a + b)^2(a + b) = (a + b)^3$$

也可以如此做

$$(a^2 + 2ab + b^2)^{\frac{3}{2}} = ((a + b)^2)^{\frac{3}{2}} = (a + b)^3$$

在下面我們要介紹 $m < n$ 時， $a^{\frac{m}{n}}$ 的計算過程，我們假設 $m < n$ ，公式 2 和 3 仍然可以用 4 我們用例子來說明

$$9^{\frac{2}{3}} = \sqrt[3]{9^2} = \sqrt[3]{81} = \sqrt[3]{27 * 3} = \sqrt[3]{3^3 * 3} = (3^3)^{\frac{1}{3}} \sqrt[3]{3} = 3\sqrt[3]{3}$$

$$16^{\frac{3}{4}} = \sqrt[4]{16^3} = \sqrt[4]{(2^4)^3} = \sqrt[3]{2^{12}} = 2^{\frac{12}{4}} = 2^3 = 8$$

我們也可以用下列的方法

$$16^{\frac{3}{4}} = \left(16^{\frac{1}{4}}\right)^3 = (2)^3 = 8$$

$$4^{\frac{2}{3}} = (4^2)^{\frac{1}{3}} = (16)^{\frac{1}{3}} = (8 * 2)^{\frac{1}{3}} = \left(8^{\frac{1}{3}}\right)\left(2^{\frac{1}{3}}\right) = (2^3)^{\frac{1}{3}}2^{\frac{1}{3}} = 2^3\sqrt[3]{2}$$

$$25^{\frac{2}{3}} = (25^2)^{\frac{1}{3}} = ((5^2)^2)^{\frac{1}{3}} = (5^4)^{\frac{1}{3}} = (5^3)^{\frac{1}{3}}5^{\frac{1}{3}} = 5^3\sqrt[3]{5}$$

$$36^{\frac{2}{3}} = (6^2)^{\frac{2}{3}} = 6^{\frac{4}{3}} = (6^4)^{\frac{1}{3}} = (6^3 * 6)^{\frac{1}{3}} = (6^3)^{\frac{1}{3}}(6)^{\frac{1}{3}} = 6^3\sqrt[3]{6}$$

$$64^{\frac{2}{3}} = (4^3)^{\frac{2}{3}} = 4^{3 * \frac{2}{3}} = 4^2 = 16$$

以下我們要討論指數為小數點的計畫，首先我們用下列的例子證明

$$2^{0.2} = 2^{\frac{1}{5}} = \sqrt[5]{2}$$

$$2^{0.3} = 2^{\frac{3}{10}} = \sqrt[10]{2^3} = \sqrt[10]{8}$$

$\sqrt[5]{2}$ 和 $\sqrt[10]{8}$ 都可以找到他們的近似值，以 $\sqrt[5]{2}$ 為例，我們假設

$$\sqrt[5]{2} = 1 + b$$

$$(1 + b)^5 = 2$$

$$1 + 5b = 2$$

$$5b = 1$$

$$b = 0.2$$

假設我們令 $b = 0.1$

$$(1.1)^5 = 1.61 \text{ 太小了}$$

我們令 $b = 1.14$

$$(1.14)^5 = 1.9255 \text{ 已經很近 2 了}$$

我們因此可知 $2^{0.2} = 0.14$