(20) 三角函數的倍角公式

我們首先要導出

$$\sin 2\theta = 2\sin\theta\cos\theta$$

我們可以用

$$\sin 2\theta = \sin(\theta + \theta)$$

$$= \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$= 2\sin\theta\cos\theta$$

我們再證

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

我們可用

$$\cos 2\theta = \cos(\theta + \theta)$$

$$= \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$= \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1 - \cos^2\theta)$$

$$= 2\cos^2\theta - 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - \sin^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

整理如下:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

(1)已知$sin\,30° = \frac{1}{2}$，求$sin\,60°$。

$$cos\,30° = \sqrt{1 - sin^2 30°}$$

$$= \sqrt{1 - \frac{1}{4}}$$

$$= \frac{\sqrt{3}}{2}$$

$$\therefore sin\,60° = 2\,sin\,30°\,cos\,30°$$

$$= 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2}$$

(2)已知$sin\,30° = \frac{1}{2}$，求$cos\,60°$。

$$cos\,60° = 1 - 2\,sin^2 30°$$

$$= 1 - 2 \times (\frac{1}{2})^2$$

$$= 1 - 2 \times \frac{1}{4}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

(3)證明$sin\,3\theta = 3\,sin\,\theta - 4\,sin^3\theta$

$$sin\,3\theta = sin(\,2\theta + \theta)$$

$$= sin\,2\theta\,cos\,\theta + cos\,2\theta\,sin\,\theta$$

$$= 2\,sin\,\theta\,cos^2\theta + (1 - 2\,sin^2\theta)\,sin\,\theta$$

$$= 2\,sin\,\theta\,(1 - sin^2\theta) + sin\,\theta - 2\,sin^3\theta$$

$$= 3\,sin\,\theta - 4\,sin^3\theta$$

(4)證明$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$
\begin{aligned}
\cos 3\theta &= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}
$$

(5)證明$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$$
\begin{aligned}
\tan 2\theta &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \\
&= \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}} \\
&= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}
$$

(6)證明$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$$
\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
&= \frac{2\sin\theta\cos\theta}{\cos^2\theta} \times \cos^2\theta \\
&= 2\tan\theta\cos^2\theta \\
&= \frac{2\tan\theta}{\sec^2\theta} \\
&= \frac{2\tan\theta}{1 + \tan^2\theta}
\end{aligned}
$$

(7)證明$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta} \times \cos^2\theta$$

$$= \frac{1-\tan^2\theta}{\sec^2\theta}$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

(8)已知$\sin\alpha = \frac{1}{\sqrt{2}}$，$\cos\beta = \frac{\sqrt{3}}{2}$，求$\sin(\alpha+\beta)$。

$\sin\alpha = \frac{1}{\sqrt{2}}$，$\cos\alpha = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\cos\beta = \frac{\sqrt{3}}{2}$，$\sin\beta = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$

$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(9)若$\sin\alpha + \sin\beta = 1$，及$\cos\alpha + \cos\beta = 0$，求$\cos 2\alpha$及$\cos 2\beta$。

$\because \sin\alpha + \sin\beta = 1$

$\sin\alpha = 1 - \sin\beta \cdots\cdots (1)$

$\because \cos\alpha + \cos\beta = 0$

$\therefore \cos\alpha = -\cos\beta$

$(1)^2+(2)^2 \to$

$$\sin^2\alpha + \cos^2\alpha = (1-\sin\beta)^2 + \cos^2\beta$$

$$= 1 - 2\sin\beta + \sin^2\beta + 1 - \sin^2\beta = 2 - 2\sin\beta$$

$\therefore 1 = 2 - 2\sin\beta$

$\sin\beta = \frac{1}{2} \cdots\cdots (3)$

$\cos 2\beta = 1 - 2\sin^2\beta$

$= 1 - 2 \times (\frac{1}{2})^2$

$= 1 - 2 \times \frac{1}{4}$

$= \frac{1}{2} \cdots\cdots (4)$

$\because \sin\alpha + \sin\beta = 1$

$\therefore \sin\alpha = 1 - \sin\beta = 1 - \frac{1}{2} = \frac{1}{2}$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$= 1 - 2 \times (\frac{1}{2})^2$

$= 1 - 2 \times \frac{1}{4}$

$= \frac{1}{2} \cdots\cdots (5)$

答案$\cos 2\alpha = \frac{1}{2}$、$\cos 2\beta = \frac{1}{2}$

我們已經學會了很多的倍角公式

現在我們可以驗證一下

(10)$\theta = 0°$

$\sin 0° = 0$，$\cos 0° = 1$

$\sin 2\theta = \sin 0° = 2\sin 0° \cos 0° = 2 \times 0 \times 1 = 0$

$\cos 2\theta = \cos 0° = 1 - 2\sin^2 0° = 1 - 2 \times 0 = 1$

(11)$\theta = 30°$

$sin\,\theta = sin\,30° = \frac{1}{2}$，$cos\,\theta = cos\,30° = \frac{\sqrt{3}}{2}$

$sin\,2\theta = sin\,60° = 2\,sin\,30°\,cos\,30° = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$cos\,2\theta = 1 - 2\,sin^2\theta = 1 - 2 \times (\frac{1}{2})^2 = 1 - 2 \times \frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}$

(12)$\theta = 45°$

$sin\,45° = \frac{1}{\sqrt{2}}$，$cos\,45° = \frac{1}{\sqrt{2}}$

$sin\,2\theta = sin\,90° = 2\,sin\,45°\,cos\,45° = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 2 \times \frac{1}{2} = 1$

$cos\,2\theta = cos\,90° = 1 - 2\,sin^2\,45° = 1 - 2 \times (\frac{1}{\sqrt{2}})^2 = 1 - 2 \times \frac{1}{2} = 1 - 1 = 0$

(13)$\theta = 60°$

$sin\,60° = \frac{\sqrt{3}}{2}$，$cos\,60° = \frac{1}{2}$

$sin\,2\theta = sin\,120° = 2\,sin\,60°\,cos\,60° = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$

$cos\,2\theta = cos\,120° = 1 - 2\,sin^2\,60° = 1 - 2 \times (\frac{\sqrt{3}}{2})^2 = 1 - 2 \times \frac{3}{4} = 1 - \frac{3}{2} = -\frac{1}{2}$

(120 度在第二象限)

(14)$\theta = 90°$

$sin\,90° = 1$，$cos\,90° = 0$

$sin\,2\theta = sin\,180° = 2\,sin\,90°\,cos\,90° = 2 \times 1 \times 0 = 0$

$cos\,2\theta = cos\,180° = 1 - 2\,sin^2\,90° = 1 - 2 \times 1^2 = 1 - 2 = -1$