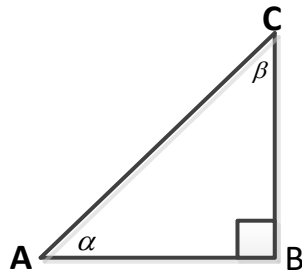


(11)三角函數的轉換

同學們一定對以下的知識很熟悉：

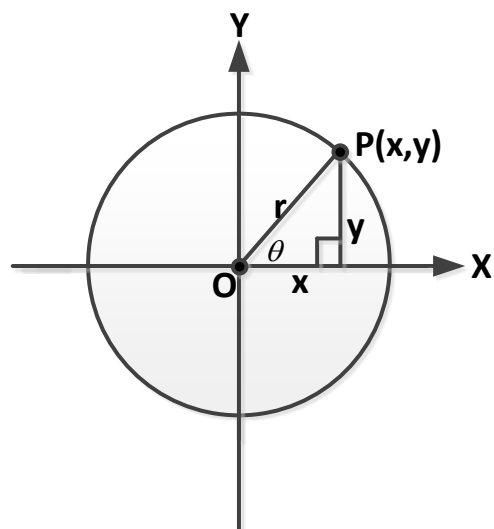


因為 $\alpha + \beta = 90^\circ$ ，所以它們有以下的關係：

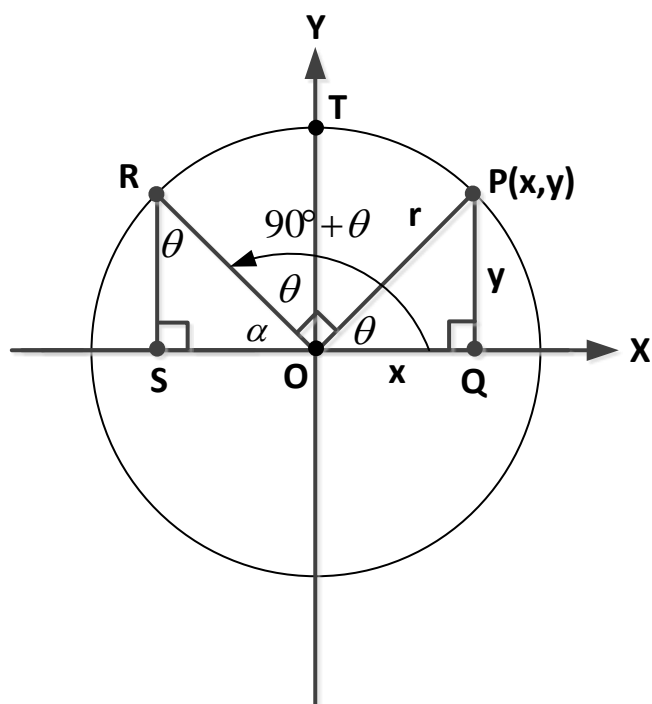
$$\sin \alpha = \frac{\overline{BC}}{\overline{AC}} = \cos \beta$$

$$\cos \alpha = \frac{\overline{AB}}{\overline{AC}} = \sin \beta$$

我們先假設 θ 是一銳角。



(1) $90^\circ + \theta$



$\angle TOR = \angle POQ = \theta$ ，且 $\angle POR = 90^\circ$ ，所以 $\angle QOR = 90^\circ + \theta$

$\because \overline{RS} \parallel \overline{OT}$ ， $\therefore \angle ORS = \theta$

考慮 $\triangle OPQ$ 和 $\triangle ORS$

$$\angle ROS + \angle POQ = 90^\circ, \text{ 而 } \angle OPQ + \angle POQ = 90^\circ$$

$$\therefore \angle ROS = \angle OPQ$$

$$\because \overline{RS} \parallel \overline{OT}, \text{ 且 } \angle ORS = \theta = \angle POQ, \overline{OR} = \overline{OP} = r$$

$$\therefore \triangle OPQ \cong \triangle ORS \text{ (ASA)}$$

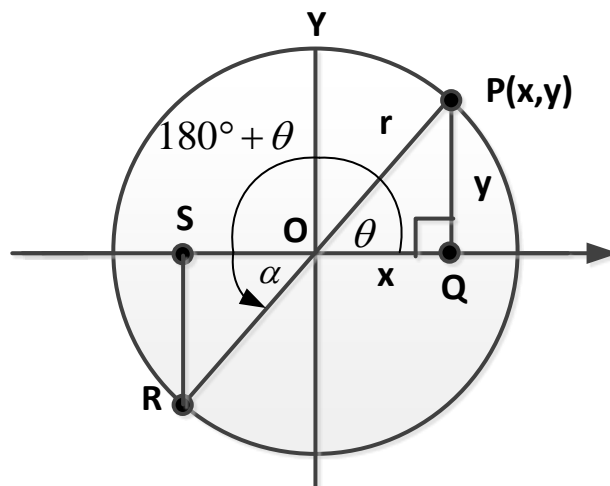
$$\therefore \overline{RS} = \overline{OQ} = x, \overline{OS} = \overline{PQ} = y$$

$$\sin(90^\circ + \theta) = \sin \alpha = \frac{\overline{RS}}{r} = \frac{x}{r} = \cos \theta$$

$$\cos(90^\circ + \theta) = \cos \alpha = \frac{\overline{OS}}{r} = \frac{-y}{x} = -\sin \theta \text{ (} x \text{ 在第二象限為負)}$$

$$\tan(90^\circ + \theta) = \tan \alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{x}{-y} = -\cot \theta$$

(2) $180^\circ + \theta$



$$\angle ROS = \alpha = \angle POQ = \theta \text{ (對頂角)}$$

$$\angle QOR = 180^\circ + \theta$$

我們可以證明 $\triangle OPQ \cong \triangle ORS$

$$\therefore \overline{OS} = \overline{OQ} = x$$

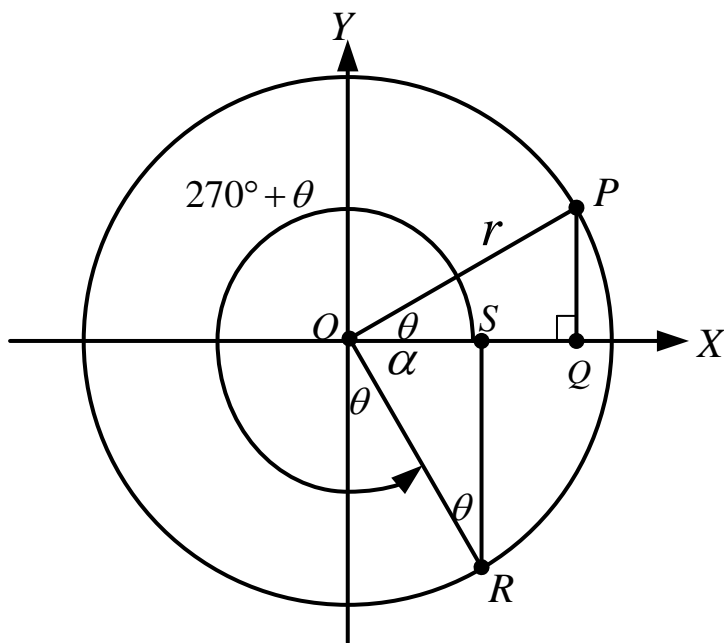
$$\overline{RS} = \overline{PQ} = y$$

$$\sin(180^\circ + \theta) = \sin \alpha = \frac{\overline{RS}}{r} = \frac{-y}{r} = -\sin \theta \quad (y \text{ 在第三象限為負})$$

$$\cos(180^\circ + \theta) = \cos \alpha = \frac{\overline{OS}}{r} = \frac{-x}{r} = -\cos \theta \quad (x \text{ 在第三象限為負})$$

$$\tan(180^\circ + \theta) = \tan \alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{-y}{-x} = \tan \theta$$

(3) $270^\circ + \theta$



$$\angle ROT = \theta$$

$$\angle POR = 270^\circ + \theta$$

我們可以證明 $\triangle OPQ \cong \triangle ORS$

$$\overline{OS} = \overline{PQ} = y, \overline{RS} = \overline{OQ} = x$$

$$\sin(270^\circ + \theta) = \sin \alpha = \frac{\overline{RS}}{r} = \frac{-x}{r} = -\cos \theta \quad (y \text{ 在第四象限為負})$$

$$\cos(270^\circ + \theta) = \cos \alpha = \frac{\overline{OS}}{r} = \frac{y}{r} = \sin \theta$$

$$\tan(270^\circ + \theta) = \tan \alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{-x}{y} = -\cot \theta$$

我們將以上的公式整理如下：

$$\sin(90^\circ + \theta) = \cos \theta$$

$$\sin(180^\circ + \theta) = -\sin \theta$$

$$\sin(270^\circ + \theta) = -\cos \theta$$

$$\cos(90^\circ + \theta) = -\sin \theta$$

$$\cos(180^\circ + \theta) = -\cos \theta$$

$$\cos(270^\circ + \theta) = \sin \theta$$

$$\tan(90^\circ + \theta) = -\cot \theta$$

$$\tan(180^\circ + \theta) = \tan \theta$$

$$\tan(270^\circ + \theta) = -\cot \theta$$

因此，我們不必將 θ 限於銳角，而使得三角函數的角可以任意象限的角，也就是所謂廣義的三角函數。

我們的結論可以用以下的表表示

| | 第 1 象限 | 第 2 象限 | 第 3 象限 | 第 4 象限 |
|---------------|--------|--------|--------|--------|
| $\sin \theta$ | + | + | - | - |
| $\cos \theta$ | + | - | - | + |
| $\tan \theta$ | + | - | + | - |

以下是一些例子

(4) $\theta = 30^\circ$

$$\sin(90^\circ + 30^\circ) = \sin 120^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin(180^\circ + 30^\circ) = \sin 210^\circ = -\sin 30^\circ = -\frac{1}{2}$$

$$\sin(270^\circ + 30^\circ) = \sin 300^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

(5) $\theta = 30^\circ$

$$\cos(90^\circ + 30^\circ) = \cos 120^\circ = -\sin 30^\circ = -\frac{1}{2}$$

$$\cos(180^\circ + 30^\circ) = \cos 210^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\cos(270^\circ + 30^\circ) = \cos 300^\circ = \sin 30^\circ = \frac{1}{2}$$

(6) $\theta = 90^\circ$

$$\sin(90^\circ + 90^\circ) = \sin 180^\circ = \cos 90^\circ = 0$$

$$\sin(180^\circ + 90^\circ) = \sin 270^\circ = -\sin 90^\circ = -1$$

$$\sin(270^\circ + 90^\circ) = \sin 360^\circ = -\cos 90^\circ = 0$$

(7) $\theta = 90^\circ$

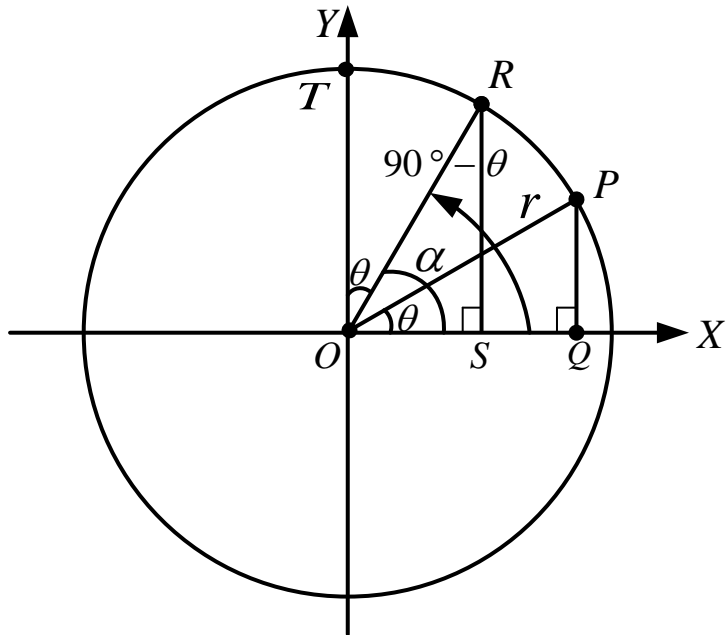
$$\cos(90^\circ + 90^\circ) = \cos 180^\circ = -\sin 90^\circ = -1$$

$$\cos(180^\circ + 90^\circ) = \cos 270^\circ = -\cos 90^\circ = 0$$

$$\cos(270^\circ + 90^\circ) = \cos 360^\circ = \sin 90^\circ = 1$$

以下我們要看另一種轉換

(8) $90^\circ - \theta$



$$\overline{PQ} = y$$

$$\overline{OQ} = x$$

$$\angle ROT = \theta$$

$$\therefore \angle ROQ = 90^\circ - \theta = \alpha$$

我們可以證明 $\triangle ROS \cong \triangle POQ$

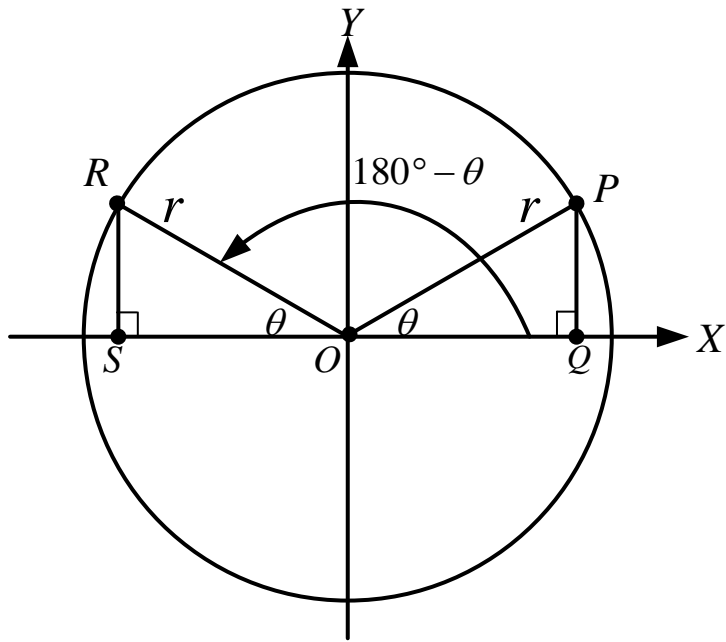
$$\overline{OR} = \overline{OQ} = r, \overline{RS} = \overline{PQ} = y$$

$$\sin(90^\circ - \theta) = \sin \alpha = \frac{\overline{RS}}{r} = \frac{y}{r} = \cos \theta$$

$$\cos(90^\circ - \theta) = \cos \alpha = \frac{\overline{OS}}{r} = \frac{x}{r} = \sin \theta$$

$$\tan(90^\circ - \theta) = \tan \alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{y}{x} = \cot \theta$$

(9) $180^\circ - \theta$



$$\overline{PQ} = y$$

$$\overline{OQ} = x$$

$$\angle POR = 180^\circ - \theta$$

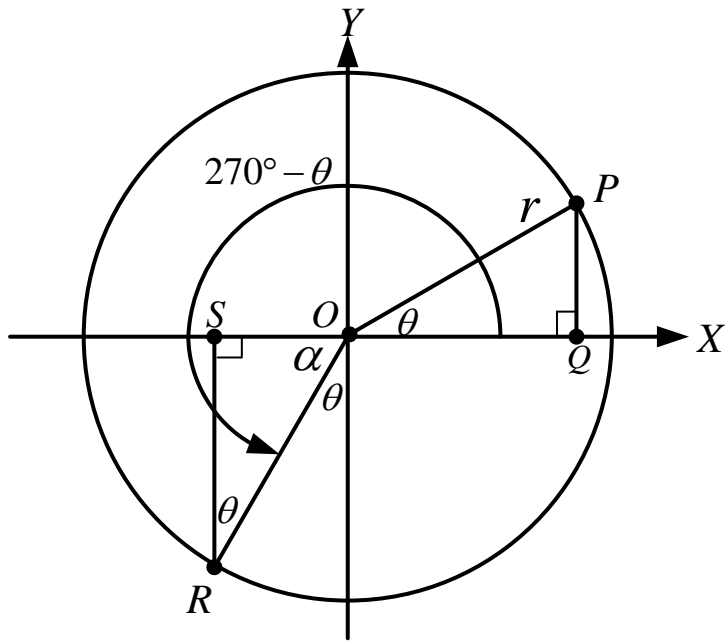
用同樣的方法，我們可以證明

$$\sin(180^\circ - \theta) = \frac{y}{r} = \sin \theta$$

$$\cos(180^\circ - \theta) = \frac{-x}{r} = -\cos \theta \text{ (} x \text{在第三象限為負)}$$

$$\tan(180^\circ - \theta) = \frac{y}{-x} = -\tan \theta$$

(10) $270^\circ - \theta$



$$\overline{PQ} = y$$

$$\overline{OQ} = x$$

$$\angle QOR = 180^\circ$$

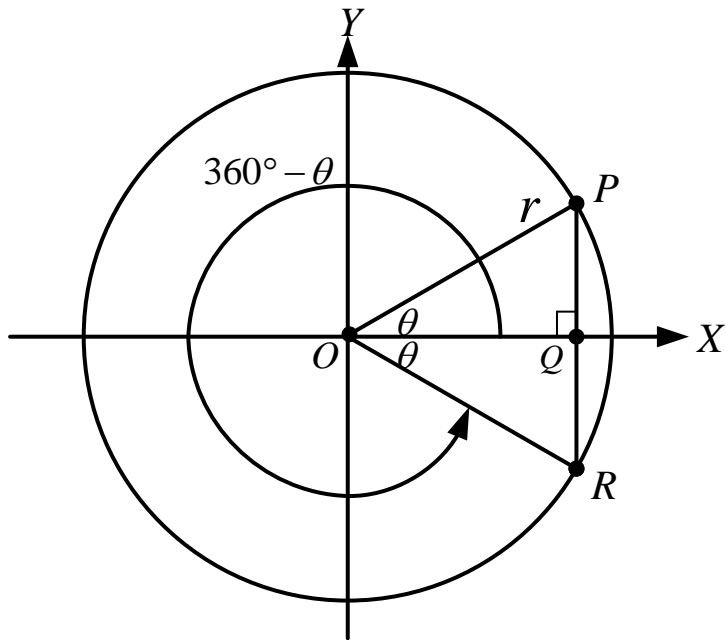
可以用同樣的方法證明

$$\sin(270^\circ - \theta) = \sin \alpha = \frac{-x}{r} = -\cos \theta \quad (y \text{ 在第三象限為負})$$

$$\cos(270^\circ - \theta) = \cos \alpha = \frac{-y}{r} = -\sin \theta \quad (x \text{ 在第三象限為負})$$

$$\tan(270^\circ - \theta) = \tan \alpha = \frac{y}{-x} = -\tan \theta$$

(11) $360^\circ - \theta$



$$\overline{PQ} = y, \overline{OQ} = x, \angle POR = 360^\circ - \theta$$

我們可以用同樣的方法證明

$$\sin(360^\circ - \theta) = \frac{-y}{r} = -\sin \theta \quad (y \text{ 在第四象限為負})$$

$$\cos(360^\circ - \theta) = \frac{x}{r} = \cos \theta$$

$$\tan(360^\circ - \theta) = \frac{-y}{x} = -\tan \theta$$

我們可以將以上的公式整理如下：

$$\sin(90^\circ - \theta) = \cos \theta$$

$$\cos(90^\circ - \theta) = \sin \theta$$

$$\sin(180^\circ - \theta) = \sin \theta$$

$$\cos(180^\circ - \theta) = -\cos \theta$$

$$\sin(270^\circ - \theta) = -\cos \theta$$

$$\cos(270^\circ - \theta) = -\sin \theta$$

$$\sin(360^\circ - \theta) = -\sin \theta$$

$$\cos(360^\circ - \theta) = \cos \theta$$

$$(12)\theta = 30^\circ$$

$$\sin(90^\circ - 30^\circ) = \sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin(180^\circ - 30^\circ) = \sin 150^\circ = \sin 30^\circ = \frac{1}{2}$$

$$\sin(270^\circ - 30^\circ) = \sin 240^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\sin(360^\circ - 30^\circ) = \sin 330^\circ = -\sin 30^\circ = -\frac{1}{2}$$

$$(13)\theta = 30^\circ$$

$$\cos(90^\circ - 30^\circ) = \cos 60^\circ = \sin 30^\circ = \frac{1}{2}$$

$$\cos(180^\circ - 30^\circ) = \cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\cos(270^\circ - 30^\circ) = \cos 240^\circ = -\sin 30^\circ = -\frac{1}{2}$$

$$\cos(360^\circ - 30^\circ) = \cos 330^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

我們可以將以上所有的公式整理如下：

$$\sin(90^\circ + \theta) = \cos \theta \dots\dots (11.1)$$

$$\sin(90^\circ - \theta) = \cos \theta \dots\dots (11.2)$$

$$\cos(90^\circ + \theta) = -\sin \theta \dots\dots (11.3)$$

$$\cos(90^\circ - \theta) = \sin \theta \dots\dots (11.4)$$

$$\sin(180^\circ + \theta) = -\sin \theta \dots\dots (11.5)$$

$$\sin(180^\circ - \theta) = \sin \theta \dots\dots (11.6)$$

$$\cos(180^\circ + \theta) = -\cos \theta \dots\dots (11.7)$$

$$\cos(180^\circ - \theta) = -\cos \theta \dots\dots (11.8)$$

$$\sin(270^\circ + \theta) = -\cos \theta \dots\dots (11.9)$$

$$\sin(270^\circ - \theta) = -\cos \theta \dots\dots (11.10)$$

$$\cos(270^\circ + \theta) = \sin \theta \dots\dots (11.11)$$

$$\cos(270^\circ - \theta) = -\sin \theta \dots\dots (11.12)$$

$$\sin(360^\circ - \theta) = -\sin \theta \dots\dots (11.13)$$

$$\cos(360^\circ - \theta) = \cos \theta \dots\dots (11.14)$$

我們可以利用以上的公式發現很多有趣的現象，以下的例子很值得同學們看的：

(14) 120°

$$120^\circ = 90^\circ + 30^\circ = 180^\circ - 60^\circ$$

根據(11.1)

$$\sin 120^\circ = \sin(90^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

根據(11.6)

$$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

(15) 210°

$$210^\circ = 180^\circ + 30^\circ = 270^\circ - 60^\circ$$

根據(11.5)

$$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

根據(11.10)

$$\sin 210^\circ = \sin(270^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$$

(16) 300°

$$300^\circ = 270^\circ + 30^\circ = 360^\circ - 60^\circ$$

根據(11.9)

$$\sin 300^\circ = \sin(270^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

根據(11.13)

$$\sin 300^\circ = \sin(360^\circ - 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$