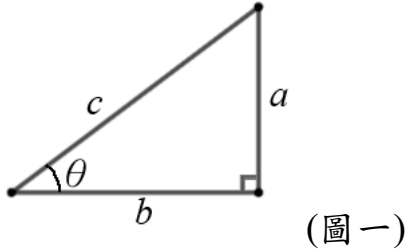


(08) 三角函數

一、基本的三角函數定義與性質

1.



如(圖一), $\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$, $\tan\theta = \frac{a}{b}$

$$\because a^2 + b^2 = c^2, c = \sqrt{a^2 + b^2}$$

$$\therefore \sin\theta = \frac{a}{\sqrt{a^2 + b^2}}$$

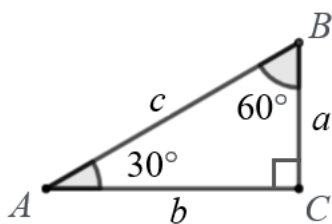
$$\cos\theta = \frac{b}{\sqrt{a^2 + b^2}}$$

2. 如(圖一), 若 $\theta = 0^\circ$, 則 $a = 0$

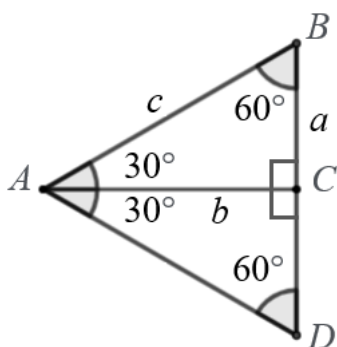
$$\sin\theta = \frac{a}{\sqrt{a^2 + b^2}} = \frac{0}{\sqrt{a^2 + b^2}} = 0$$

$$\cos\theta = \frac{b}{\sqrt{a^2 + b^2}} = \frac{b}{\sqrt{0 + b^2}} = 1$$

3. 若 $\theta = 30^\circ$ ，如何得到 $\sin 30^\circ$ 、 $\cos 30^\circ$ 、 $\tan 30^\circ$ ？



再加上一個相同的三角形



合成的 $\triangle ABD$ ，因為三內角都是 60° ，所以 $\triangle ABD$ 是正三角形

$$\therefore a = \frac{1}{2}BD = \frac{1}{2}AB = \frac{1}{2}c, \quad c = 2a$$

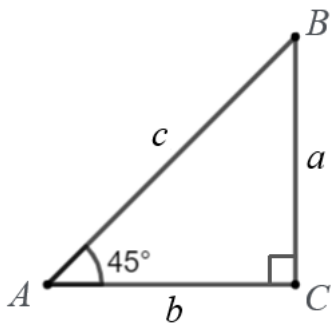
$$b^2 = c^2 - a^2 = (2a)^2 - a^2 = 4a^2 - a^2 = 3a^2, \quad b = \sqrt{3}a \text{ (長度不取負值)}$$

$$\sin 30^\circ = \frac{a}{c} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{b}{c} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{a}{b} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

4. 若 $\theta = 45^\circ$ ，如何得到 $\sin 45^\circ$ 、 $\cos 45^\circ$ 、 $\tan 45^\circ$ ？



$\triangle ABC$ 是等腰直角三角形， $a = b$

$$c^2 = a^2 + b^2 = a^2 + a^2 = 2a^2$$

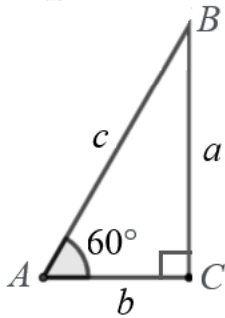
$$c = \sqrt{2}a \text{ (長度不取負值)}$$

$$\sin 45^\circ = \frac{a}{c} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

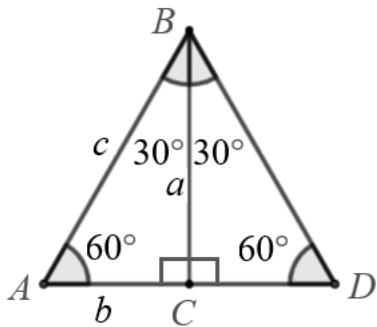
$$\cos 45^\circ = \frac{b}{c} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{a}{b} = \frac{a}{a} = 1$$

5. 若 $\theta = 60^\circ$ ，如何得到 $\sin 60^\circ$ 、 $\cos 60^\circ$ 、 $\tan 60^\circ$ ？



再加上一個相同的三角形



合成的 $\triangle ABD$ ，因為三內角都是 60° ，所以 $\triangle ABD$ 是正三角形

$$c = \overline{AB} = \overline{AD} = 2b, \quad b = \frac{1}{2}c$$

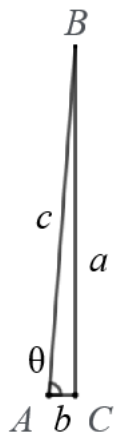
$$a^2 = c^2 - b^2 = c^2 - \left(\frac{1}{2}c\right)^2 = c^2 - \frac{1}{4}c^2 = \frac{3}{4}c^2, \quad a = \frac{\sqrt{3}}{2}c \quad (\text{長度不取負值})$$

$$\sin 60^\circ = \frac{a}{c} = \frac{\frac{\sqrt{3}}{2}c}{c} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{b}{c} = \frac{\frac{1}{2}c}{c} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{a}{b} = \frac{\frac{\sqrt{3}}{2}c}{\frac{1}{2}c} = \sqrt{3}$$

6. 若 $\theta = 90^\circ$ ，如何得到 $\sin 90^\circ$ 、 $\cos 90^\circ$ ？



我們可以想像 θ 越來越接近 90° 時， b 會接近 0 ， a 會接近 c 。

當 $\theta = 90^\circ$ ， $b = 0$ ， $a = c$

$$\sin 90^\circ = \frac{a}{c} = \frac{c}{c} = 1$$

$$\cos 90^\circ = \frac{b}{c} = \frac{0}{c} = 0$$

7. 除了 \sin 、 \cos 、 \tan 以外，還有 \csc 、 \sec 、 \cot 。

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

8. 重要角度的三角函數

角度單位除了度($^{\circ}$)以外，還有弧度

常用的角度與弧度對應如下

$$360^{\circ} = 2\pi, 180^{\circ} = \pi, 90^{\circ} = \frac{1}{2}\pi, 60^{\circ} = \frac{1}{3}\pi, 45^{\circ} = \frac{1}{4}\pi, 30^{\circ} = \frac{1}{6}\pi$$

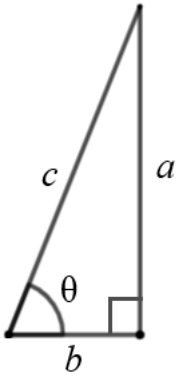
θ (角度)	0°	30°	45°	60°	90°
θ (弧度)	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞ (無限大)

當 θ 的範圍介於 $0^{\circ} \leq \theta \leq 90^{\circ}$ 時，我們可以看出以下幾點：

- (1) θ 越大， $\sin\theta$ 越大。
- (2) θ 越大， $\cos\theta$ 越小。
- (3) θ 越大， $\tan\theta$ 越大。

二、三角函數的應用

9. 下圖為直角三角形，若 $a=5$ ， $b=2$ 。求 $\sin\theta$ 、 $\cos\theta$ 、 $\tan\theta$ 。



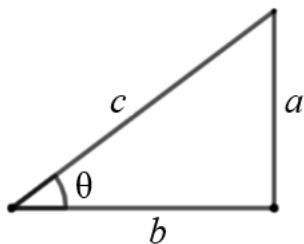
$$c = \sqrt{a^2 + b^2} = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}$$

$$\sin\theta = \frac{a}{c} = \frac{5}{\sqrt{29}}$$

$$\cos\theta = \frac{b}{c} = \frac{2}{\sqrt{29}}$$

$$\tan\theta = \frac{a}{b} = \frac{5}{2}$$

10. 已知 $\sin\theta = 0.6$ ，求 $\cos\theta$ 和 $\tan\theta$ 。



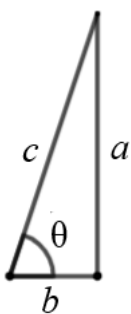
$$\sin\theta = \frac{a}{c} = 0.6, a = 0.6c$$

$$b = \sqrt{c^2 - a^2} = \sqrt{c^2 - (0.6c)^2} = \sqrt{c^2 - 0.36c^2} = \sqrt{0.64c^2} = 0.8c$$

$$\cos\theta = \frac{b}{c} = \frac{0.8c}{c} = 0.8$$

$$\tan\theta = \frac{a}{b} = \frac{0.6c}{0.8c} = \frac{6}{8} = \frac{3}{4}$$

11. 已知 $\tan\theta = 3$ ，求 $\sin\theta$ 和 $\cos\theta$ 。



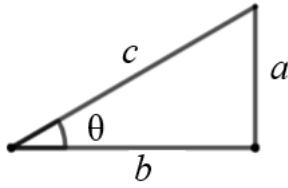
$$\tan\theta = \frac{a}{b} = 3, a = 3b$$

$$c = \sqrt{a^2 + b^2} = \sqrt{(3b)^2 + b^2} = \sqrt{9b^2 + b^2} = \sqrt{10b^2} = \sqrt{10}b$$

$$\sin\theta = \frac{a}{c} = \frac{3b}{\sqrt{10}b} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

$$\cos\theta = \frac{b}{c} = \frac{b}{\sqrt{10}b} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$

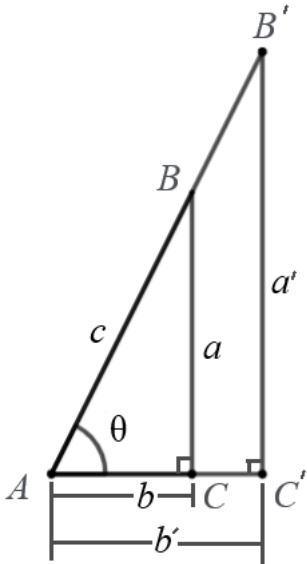
12. 已知 $\sin\theta = 0.5$ ， $a = 3$ 求 b 。



$$\sin\theta = \frac{a}{c} = 0.5, \quad c = 2a$$

$$b = \sqrt{c^2 - a^2} = \sqrt{(2a)^2 - a^2} = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = \sqrt{3}a = \sqrt{3} \times 3 = 3\sqrt{3}$$

13. 如下圖， $\triangle ABC$ 中， $\tan\theta = 2$ 。從 \overline{AC} 延長線取一點 C' ，從 \overline{AB} 延長線取一點 B' ，且 $\overline{AC} \perp \overline{AB}$ ，求 $\frac{a}{b}$ 。



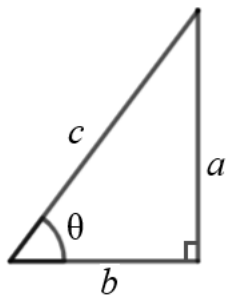
$$\because \angle BAC = \angle B'AC', \quad \angle ACB = \angle AC'B'$$

$\therefore \triangle ABC$ 與 $\triangle AB'C'$ 相似 (AA 相似)

$$\frac{a}{b} = \frac{a'}{b'} \quad (\text{相似三角形對應邊成比例})$$

$$\frac{a'}{b'} = \frac{a}{b} = \tan\theta = 2$$

14. 如下圖， $\tan\theta = \frac{4}{3}$ ，求 $\sin\theta$ 和 $\cos\theta$ 。



$$\sin\theta = \frac{a}{b} = \frac{4}{3}, \quad a = \frac{4}{3}b$$

$$c = \sqrt{a^2 + b^2} = \sqrt{\left(\frac{4}{3}b\right)^2 + b^2} = \sqrt{\frac{16}{9}b^2 + b^2} = \sqrt{\frac{25}{9}b^2} = \frac{5}{3}b$$

$$\sin\theta = \frac{a}{c} = \frac{\frac{4}{3}b}{\frac{5}{3}b} = \frac{4}{5}$$

$$\cos\theta = \frac{b}{c} = \frac{b}{\frac{5}{3}b} = \frac{3}{5}$$