(73)等比級數

若$a_1, a_2, \cdots\cdots, a_n$滿足以下的條件，則此數列是一個等比級數。

$a_1 = a$

$a_2 = ar$

$a_3 = ar^2$

$\vdots$

$a_n = ar^{n-1}$

例

$a_1 = 1, r = 2$，則$1,2,4,8,16,32,64$為一等比級數

$a_1 = -1, r = 2$，則$-1, -2, -4, -8, -16, -32, -64$為一等比級數

$a_1 = -1, r = -2$，則$-1,2,-4,8,-16,32,-64$為一等比級數

1. $a = 3,\ r = 2,$ 求a_5

   $a_5 = ar^{n-1} = 3(2)^4 = 3 \times 16 = 48$

2. $a = -3,\ r = -2,$ 求a_5

   $a_5 = ar^{n-1} = (-3)(-2)^4 = (-3) \times 16 = -48$

3. $a_1 = -3,\ r = -2,$ 求a_6

   $a_6 = ar^{n-1} = (-3)(-2)^5 = (-3)(-32) = 96$

4. $a = 2,\ r = \frac{1}{2},$ 求a_5

   $$a_5 = ar^{n-1} = 2(\frac{1}{2})^4 = 2 \times \frac{1}{16} = \frac{1}{8}$$

5. $a = 1,\ r = \sqrt{2},$ 求a_5

   $a_5 = ar^{n-1} = (1)\left(\sqrt{2}\right)^4 = 4$

6. $a = 1,\ r = \sqrt{2},$ 求a_6

   $a_6 = ar^{n-1} = (1)\left(\sqrt{2}\right)^5 = 4\sqrt{2}$

7. 已知$r = 2, a_5 = 48$，求a

$a_5 = ar^{n-1} = a(2)^4 = 48$

$16a = 48$

$a = 3$

8. $r = -2, a_6 = 96$，求a

$a_6 = ar^{n-1} = a(-2)^5 = 96$

$-32a = 96$

$a = -3$

9. $\mathrm{r} = \frac{1}{2}$，$a_5 = \frac{1}{8}$，求a

$a_5 = ar^{n-1} = a(\frac{1}{2})^4 = \frac{1}{8}$

$\frac{1}{16}a = \frac{1}{8}$

$a = 2$

10. $r = \sqrt{2}$，$a_5 = 4$， 求 a

$a_5 = ar^{n-1} = a(\sqrt{2})^4 = 4$

$4a = 4$

$a = 1$

11. $a = 3$，$a_5 = 48$, 求 r

$a_5 = ar^{n-1} = 3(r)^4 = 48$

$(r)^4 = 16$

$\mathrm{r} = \pm 2, \pm 2\mathrm{i}$

12. $a = -3$，$a_5 = -48$, 求 r

$a_5 = ar^{n-1} = (-3)(r)^4 = -48$

$(r)^4 = 16$

$\mathrm{r} = \pm 2, \pm 2\mathrm{i}$

13. $a = -3$，$a_6 = 96$, 求 r

$a_6 = ar^{n-1} = (-3)(r)^5 = 96$

$(r)^5 = -32$

$\mathrm{r} = -2$

因為$a_i = \mathrm{a}r^{i-1}, a_j = ar^{j-1}$

故$\frac{a_j}{a_i} = \frac{ar^{j-1}}{\mathrm{a}r^{i-1}} = r^{(j-1)-(i-1)} = r^{j-i}$

∴ $a_j = a_i r^{j-i}$

14. 已知$a_5 = 48,\ r = 2,$ 求a_7
    $a_7 = a_5 r^{7-5} = (48)(2)^2 = 192$

15. 已知$a_5 = -48,\ r = -2,$ 求a_7
    $a_7 = a_5 r^{7-5} = (-48)(-2)^2 = -192$

16. 已知$a_6 = 96,\ r = -2,$ 求a_2
    $a_6 = a_2 r^{6-2} = a_2 r^4 = a_2(-2)^4 = 96$
    $a_2(16) = 96$
    $a_2 = 6$

17. 已知$a_6 = 96,\ r = -2,$ 求a_1
    $a_6 = a_1 r^{6-1} = a_1 r^5 = a_1(-2)^5 = 96$
    $a_2(-32) = 96$
    $a_2 = -3$

等比級數還有一個有趣的公式:
如a, b, c為等比級數，則$b^2 = ac$
證明: $b = ar,\ c = ar^2$
$\mathrm{a}c = a(ar^2\ ) = a^2 r^2 = (ar)^2 = b^2$

18. a, b, c為等比級數，$a = 2, c = 8$，求b
    $b^2 = ac = 2 \times 8 = 16$
    $b = \pm 4$

19. a, b, c為等比級數，$a = -3, c = -27$，求 b
    $b^2 = ac = (-3) \times (-27) = 81$
    $b = \pm 9$

20. a, b, c為等比級數，$a = -2, c = -8$，求r

$b^2 = ac = (-2) \times (-8) = 16$

$b = \pm 4$

$b = ar$

$\pm 4 = -2r$

$r = \pm 2$

21. 已知$a = 1, r = -3, S_n = -20$，求n

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{(1)((-3)^n - 1)}{-3 - 1} = -20$$

$$\frac{(-3)^n - 1}{-4} = -20$$

$(-3)^n - 1 = 80$

$(-3)^n = 81$

$\mathrm{n} = 4$

驗證$S_n = 1 + (-3) + 9 + (-27) = -20$

22. 已知 $r = 3, S_4 = 80$，求a

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a((3)^4 - 1)}{3 - 1} = 80$$

$$\frac{80a}{2} = 80$$

$80a = 160$

$a = 2$

驗證$S_4 = 2 + 6 + 18 + 54 = 80$

23. 已知 $r = -2, S_5 = -11$，求a

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a((-2)^5 - 1)}{-2 - 1} = -11$$

$$\frac{-33a}{-3} = -11$$

$-33a = 33$

$a = -1$

驗證$S_5 = -1 + 2 + (-4) + 8 + (-16) = -11$

24. a, b, c為一等比級數，$a = x^2 + 4, c = 1, b^2 = 4x$，求$a, b$

$ac = b^2$

$x^2 + 4 = 4x$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x = 2$

$a = x^2 + 4 = 2^2 + 4 = 8$

$b^2 = 4x = 8$

$\mathrm{b} = \pm\sqrt{8} = \pm 2\sqrt{2}$

此等比級數的 r 是

$$\mathrm{r} = \frac{b}{a} = \frac{\pm 2\sqrt{2}}{8} = \pm\frac{\sqrt{2}}{4}$$

驗證

$a = 8$

$$b = 8r = 8 \times (\pm\frac{\sqrt{2}}{4}) = \pm 2\sqrt{2}$$

$$c = 8r^2 = 8(\pm\frac{\sqrt{2}}{4})^2 = 8\left(\frac{2}{16}\right) = 1$$

答案是正確的

25. a, b, c為一等比級數，$a = x^2 - 4, c = 1, b^2 = 3x$，求$a, b$

$ac = b^2$

$x^2 - 4 = 3x$

$x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

$x = 4$及$x = -1$

若$x = 4$

$a = x^2 - 4 = 4^2 - 4 = 12$

$b^2 = 3x = 12$

$b = \pm\sqrt{12} = \pm 2\sqrt{3}$

$$\mathrm{r} = \frac{b}{a} = \frac{\pm 2\sqrt{3}}{12} = \pm\frac{\sqrt{3}}{6}$$

可以證明$a = 12, b = \pm 2\sqrt{3}, c = 1$為一等比級數

若$x = -1$

$a = x^2 - 4 = (-1)^2 - 4 = -3$

$b^2 = 3x = -3$

b=$\sqrt{3}i$

$$\mathrm{r} = \frac{b}{a} = \frac{\sqrt{3}i}{-3} = \frac{-i}{\sqrt{3}}$$

此時$c = br = (\sqrt{3}i)(\frac{-i}{\sqrt{3}}) = -i^2 = 1$

可以證明$a = -3, b = \sqrt{3}i, c = 1$為一等比級數

26. a, b, c為一等比級數，$a = x^2 + 6,\ b^2 = 5x,\ c = 1$，求$a, b$

$ac = b^2$

$x^2 + 6 = 5x$

$x^2 - 5x + 6 = 0$

$(x - 3)(x - 2) = 0$

$x = 3$及$x = 2$

若$x = 3$

$a = x^2 + 6 = 3^2 + 6 = 15$

$b^2 = 5x = 15$

$b = \pm\sqrt{15}$

$$\mathrm{r} = \frac{b}{a} = \frac{\pm\sqrt{15}}{15} = \pm\frac{1}{\sqrt{15}}$$

可以證明$a = 15, b = \pm\sqrt{15}, c = 1$為一等比級數

若$x = 2$

$a = x^2 + 6 = 2^2 + 6 = 10$

$b^2 = 5x = 10$

$b = \pm\sqrt{10}$

$$\mathrm{r} = \frac{b}{a} = \frac{\pm\sqrt{10}}{10} = \pm\frac{1}{\sqrt{10}}$$

可以證明$a = 10, b = \pm\sqrt{10}, c = 1$為一等比級數

27. a, b, c為一等比級數，$a = x - 3,\ \ b^2 = 4,\ c = x$，求$a, b, c$

$ac = b^2$

$x(x - 3) = 4$

$x^2 - 3x - 4 = 0$

$(x + 1)(x - 4) = 0$

$x = -1$ 及$x = 4$

若$x = -1$

$a = x - 3 = -1 - 3 = -4$

$b^2 = 4$

$b = 2$

$c = -1$

$$r = \frac{b}{a} = \frac{2}{-4} = -\frac{1}{2}$$

$-4, 2, -1$為一等比級數

若$x = 4$

$a = x - 3 = 4 - 3 = 1$

$b^2 = 4$

$b = 2$

$c = 4$

$$r = \frac{b}{a} = \frac{2}{1} = 2$$

1,2,4為一等比級數

28. a, b, c為一等差級數，也是一等比級數，$abc = 27$，求a, b, c

$\because a, b, c$為一等比級數，故

$ac = b^2$

$\therefore abc = b^2(b) = b^3 = 27$

$\therefore b = 3$

a, b, c為一等差級數，故

$a + c = 2b$

$a = 2b - c = 6 - c$

$\because abc = 27, b = 3$

$(6 - c)3c = 27$

$c^2 - 6\mathrm{c} + 9 = 0$

$(c - 3)^2 = 0$

$\mathrm{c} = 3$

$a = 6 - c = 3$

在以下，我們要討論等比級數的和

令$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$

$rs_n = \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$s_n - rs_n = (1 - r)s_n = a - ar^n = a(1 - r^n)$

$\therefore s_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$

29. $n = 5,\ a = 1,\ r = 2,$ 求S_n

$S_n = 1 + 2 + 4 + 8 + 16 = 31$

用公式$s_n = \frac{a(r^n-1)}{r-1} = \frac{1(2^5-1)}{2-1} = 31$

也可以用$s_n = \frac{a(1-r^n)}{1-r}$，可以得到相同的結果

30. $n = 5,\ a = -1,\ r = -2,$ 求S_n

$S_n = -1 + 2 + (-4) + 8 + (-16) = -11$

用公式$s_n = \frac{a(r^n-1)}{r-1} = \frac{(-1)((-2)^5-1)}{-2-1} = \frac{(-1)(-32-1)}{-3} = \frac{(-1)(-33)}{-3} = -11$