(72)等差級數

如果$a_1, a_2, \cdots\cdots, a_n$是一個等差級數，則

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = a_1 + 2d$

$\vdots$

$a_n = a_1 + (n-1)d$

a_1是首項，d 是公差。

例

$a_1 = 1, d = 1$，則$1,2,3,\dots,100$是一個等差級數

$a_1 = 0, d = -2$，則$0,-2,-4,\dots,-100$是一個等差級數

$a_1 = 4, d = -1$，則$3,2,1,0,-1,-2,\dots,-100$是一個等差級數

$a_1 = 1, d = \frac{1}{2}$，則$1,\frac{3}{2},\frac{4}{2},\frac{5}{2},\frac{6}{2},\dots,\frac{100}{2}$是一個等差級數

1. $a_1 = 2,\ a_{10} = 20$，求d

   $a_{10} = a_1 + (10-1)d$

   $20 = 2 + 9d$

   $9d = 18$

   $d = 2$

2. $a_4 = 5,\ a_6 = 1$，求d

   $a_6 = a_4 + (6-4)d$

   $1 = 5 + 2d$

   $-4 = 2d$

   $d = -2$

3. $a_5 = -6,\ d = -2$，求a_1

   $a_5 = a_1 + (5-1)d$

   $-6 = a_1 + 4(-2)$

   $-6 = a_1 - 8$

   $a_1 = -6 + 8$

   $a_1 = 2$

4. $a_5 = 3,\ d = 2$，求a_7
$a_7 = a_5 + (7-5)d$
$a_7 = 3 + 2(2)$
$a_7 = 7$

5. $a_1 = -4,\ d = 2,\ a_n = 4$，求 n
$a_n = a_1 + (n-1)d$
$4 = -4 + 2(n-1)$
$2n - 2 - 4 = 4$
$2n = 4 + 6 = 10$
$n = 5$

6. $a_5 = 2,\ d = 3,\ a_n = 11$，求 n
$a_n = a_5 + (n-5)d$
$11 = 2 + 3(n-5)$
$11 = 2 + 3n - 15$
$11 = 3n - 13$
$3n = 24$
$n = 8$

7. 試證$a_i + a_{i+2} = 2a_{i+1}$
$a_i = a_1 + (i-1)d$
$a_{i+2} = a_1 + (i+2-1)d = a_1 + (i+1)d$
$$\begin{aligned} a_i + a_{i+2} &= a_1 + (i-1)d + a_1 + (i+1)d \\ &= 2a_1 + (i-1+i+1)d \\ &= 2a_1 + 2id \\ &= 2(a_1 + id) \\ &= 2a_{i+1} \end{aligned}$$

8. $x^2, x, 1$為一等差級數，求x
根據$a_i + a_{i+2} = 2a_{i+1}$
故$x^2 + 1 = 2x$
$x^2 - 2x + 1 = 0$
$(x-1)^2 = 0$
$x = 1$
此題的等差級數為1,1,1，此時$d = 0$

9. $x^2, 2x, 3$為一等差級數，求x

$x^2+3=2(2x)$

$x^2-4x+3=0$

(x-1)(x-3)=0

$x=1$

$x=3$

若$x=3$，此等差級數為9,6,3，此時$d=-3$

若$x=1$，此等差級數為1,2,3，此時$d=1$

10. $\sqrt{x}, 0, -1$為一等差級數，求x

$\sqrt{x}+(-1)=0$

$\sqrt{x}=1$

$x=1$

此等差級數為1,0,−1，此時$d=-1$

11. $x^2, 2x, -32$為一等差級數，求x

$x^2-32=2(2x)$

$x^2-4x-32=0$

$(x+4)(x-8)=0$

$x=-4$

$x=8$

若$x=-4$，此等差級數為16,−8,−32，此時$d=-24$

若$x=8$，此等差級數為64,16,−32，此時$d=-48$

12. 試證$a_i-a_j=(i-j)d$

$a_i=a_1+(i-1)d$

$a_j=a_1+(j-1)d$

$a_i-a_j=(i-1-j+1)d=(i-j)d$

13. 已知$a_3=10,\ a_6=19$，求 d

$a_6-a_3=19-10=9$

$9=(6-3)d$

$9=3d$

$d=3$

14. 已知$a_3 = 10,\ a_n = 2, \mathrm{d} = -1$，求 n

$a_n = a_3 + (n-3)d$

$= 10 + (n-3)(-1)$

$= 10 - n + 3$

$= 13 - n$

$2 = 13 - n$

$n = 11$

15. 試證$a_i + a_j = a_{i+1} + a_{j-1}$

$a_i = a_1 + (i-1)d$

$a_j = a_1 + (j-1)d$

$a_{i+1} = a_1 + (i+1-1)d = a_1 + id$

$a_{j-1} = a_1 + (j-1-1)d = a_1 + (j-2)d$

$\therefore\ a_i + a_j = (a_1 + (i-1)d) + (a_1 + (j-1)d) = 2a_1 + (i+j-2)d$

$a_{i+1} + a_{j-1} = a_1 + id + (a_1 + (j-2)d) = 2a_1 + (i+j-2)d$

$\therefore a_i + a_j = a_{i+1} + a_{j-1}$

$a_i + a_j = a_{i+1} + a_{j-1}$可以用以下的圖表示

$a_i, a_{i+1}, \cdots\cdots, a_{j-1}, a_j$

請看以下的等差級數

a_1	a_2	a_3	a_4	a_5	a_6
1	3	5	7	9	11

$a_1 + a_6 = 1 + 11 = 12$

$a_2 + a_5 = 3 + 9 = 12$

$a_3 + a_4 = 5 + 7 = 12$

令$S_n = a_1 + a_2 + \cdots\cdots + a_n$

求S_n可以用以下的方法

$S_n = a_1 + a_2 + \cdots\cdots + a_n$

$S_n = a_n + a_{n-1} + \cdots\cdots + a_1$

$\therefore 2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \cdots\cdots + (a_n + a_1)$

但$(a_1+a_n)=(a_2+a_{n-1})=\cdots\cdots=(a_n+a_1)$

$\therefore 2S_n=n(a_1+a_n)$

$\therefore S_n=\frac{n}{2}(a_1+a_n)$

16. $a_1=3,\ d=3$，求S_7

    $a_7=a_1+(7-1)d=3+6\times 3=21$

    $S_7=\frac{7}{2}(a_1+a_7)=\frac{7}{2}(3+21)=84$

    驗證:$S_7=3+6+9+12+15+18+21=84$

17. 已知$S_6=42, a_1=2$，求 d

    $S_6=\frac{6}{2}(2+a_6)=42$

    $3(2+a_6)=42$

    $2+a_6=14$

    $a_6=12$

    $a_6=a_1+(6-1)d$

    $12=2+5d$

    $10=5d$

    $d=2$

18. 已知$S_6=12, a_6=7$，求a_1

    $S_6=\frac{6}{2}(a_1+7)=12$

    $3(a_1+7)=12$

    $a_1+7=4$

    $a_1=-3$

19. 承上題，求d

    $a_6=a_1+(6-1)d$

    $7=-3+5d$

    $10=5d$

    $d=2$

求S_n還有一個公式

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$\because a_n = a_1 + (n-1)d$$

$$\therefore S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d)$$

$$= \frac{n}{2}(2a_1 + (n-1)d)$$

20. 已知$a_1 = -2,\ d = -3,\ n = 6,$求$S_n$

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

$$= \frac{6}{2}\big(2(-2) + (6-1)(-3)\big)$$

$$= 3(-4-15)$$

$$= 3(-19)$$

$$= -57$$

驗證$a_6 = a_1 + (n-1)d = -2 + 5(-3) = -17$

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{6}{2}(-2 + (-17)) = 3(-19) = -57$$

利用$S_n = \frac{n}{2}(2a_1 + (n-1)d)$，我們可以導出以下的公式

$$2S_n = n(2a_1 + (n-1)d) = 2na_1 + n(n-1)d$$

$$\therefore 2na_1 = 2S_n - n(n-1)d$$

$$a_1 = \frac{2S_n - n(n-1)d}{2n}$$

21. 承上題，$S_n = -57,\ d = -3,\ n = 6$

$$\therefore a_1 = \frac{2S_n - n(n-1)d}{2n}$$
$$= \frac{2(-57) - 6(6-1)(-3)}{2 \times 6}$$
$$= \frac{-114 - (-90)}{12}$$
$$= \frac{-114 + 90}{12}$$
$$= \frac{-24}{12}$$
$$= -2$$

答案正確

我們也可以得到一個求d的公式

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$
$$2S_n = n(n-1)d + 2na_1$$
$$n(n-1)d = 2S_n - 2na_1$$
$$\therefore d = \frac{2S_n - 2na_1}{n(n-1)}$$

22. 承(20)題，$S_n = -57,\ n = 6,\ a_1 = -2$

$$d = \frac{2S_n - 2na_1}{n(n-1)}$$
$$= \frac{2(-57) - 2(6)(-2)}{6(6-1)}$$
$$= \frac{-114 + 24}{30}$$
$$= \frac{-90}{30}$$
$$= -3$$

答案是正確的，我們可以有一個求n的公式

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$
$$2S_n = n(n-1)d + 2na_1$$
$$n^2d - nd + 2na_1 - 2S_n = 0$$

23. 承(20)題，$d=-3,\ a_1=-2,\ S_n=-57$，求 n

$\therefore n^2d-nd+2na_1-2S_n=0$

$n^2(-3)-n(-3)+2n(-2)-2(-57)=0$

$-3n^2+3n-4n+114=0$

$-3n^2-n+114=0$

$3n^2+n-114=0$

$(3n+19)(n-6)=0$

$\therefore n=6$

答案是正確的

24. 承(16)題，已知$S_7=84,\ d=3$，求a_1

$$a_1=\frac{2S_n-n(n-1)d}{2n}$$

$$=\frac{2(84)-7(7-1)3}{2(7)}$$

$$=\frac{168-126}{14}$$

$$=\frac{42}{14}$$

$$=3$$

答案正確

25. 承(16)題，已知$S_7=84, a_1=3$，求 d

$$d=\frac{2S_n-2na_1}{n(n-1)}$$

$$=\frac{2(84)-2(7)(3)}{7(7-1)}$$

$$=\frac{168-42}{42}$$

$$=\frac{126}{42}$$

$$=3$$

答案正確

26. 承(16)題，已知$S_n = 84, a_1 = 3,\ d = 3$，求 n

$n^2d - nd + 2na_1 - 2S_n = 0$

$n^2(3) - 3n + 2n(3) - 2(84) = 0$

$3n^2 + 3n - 168 = 0$

$n^2 + n - 56 = 0$

$(n+8)(n-7) = 0$

$\therefore n = 7$

答案正確

我們都以為S_n不是遞減，就是遞增，其實S_n會回到a_1的，請看以下的例子:

$a_1 = -5, d = 2, S_1 = -5$

$a_2 = -5 + 2 = -3, S_2 = -5 + (-3) = -8$

$a_3 = -3 + 2 = -1, S_3 = -8 + (-1) = -9$

$a_4 = -1 + 2 = 1, S_4 = -9 + 1 = -8$

$a_5 = 1 + 2 = 3, S_5 = -8 + 3 = -5 = a_1$

我們可以推導一個公式，使我們知道$S_n = a_1$的條件

$S_n = \frac{n}{2}(2a_1 + (n-1)d) = a_1$

$n(2a_1 + (n-1)d) = 2a_1$

$2a_1n - 2a_1 + n(n-1)d = 0$

$2a_1(n-1) = -n(n-1)d$

$\therefore 2a_1 = -nd$

如$a_1 = -5,\ d = 2$，則當 n = 5 時，$S_5 = a_1$

因為$2a_1 = 2(-5) = -10$

$-nd = -5(2) = -10$

$\therefore 2a_1 = -nd$

27. $a_1 = 4,\ d = -2$，求何時$S_n = a_1 = 4$

$2a_1 = -nd$時$S_n = a_1$

$\therefore 2(4) = -n(-2)$

$n = 4$

我們可以看一下是否真的如此

$a_1 = 4, \mathrm{d} = -2, S_1 = 4$

$a_2 = 4 - 2 = 2, S_2 = 4 + 2 = 6$

$a_3 = 2 - 2 = 0, S_3 = 6 + 0 = 6$

$a_4 = 0 - 2 = -2, S_4 = 6 - 2 = 4$

的確，當$n = 4$時，$S_4 = a_1$

28. $a_2 = -6,\ d = 3$，求何時$S_n = a_1$

$2a_1 = -nd$

$2(-6) = -n(3)$

$-12 = -3n$

$n = 4$

驗證

$a_1 = -6, S_1 = -6$

$a_2 = -6 + 3 = -3, S_2 = -6 + (-3) = -9$

$a_3 = -3 + 3 = 0, S_3 = -9 + 0 = -9$

$a_4 = 0 + 3 = 3, S_4 = -9 + 3 = -6 = a_1$

∴答案是正確的

我們也應該知道S_n可能等於0的

$$S_n = \frac{n}{2}(2a_1 + (n-1)d) = 0$$

$\therefore 2a_1 + (n-1)d = 0$

$2a_1 = -(n-1)d$

29. $a_1 = -6,\ d = 3$，求何時$S_n = 0$

$2a_1 = -(n-1)d$

$2(-6) = -(n-1)3$

$-12 = -3n + 3$

$3n = 15$

$n = 5$

驗證

$a_1 = -6, S_1 = -6$

$a_2 = -6 + 3 = -3, S_2 = -6 + (-3) = -9$

$a_3 = -3 + 3 = 0, S_3 = -9 + 0 = -9$

$a_4 = 0 + 3 = 3, S_4 = -9 + 3 = -6$

$a_5 = 3 + 3 = 6, S_4 = -6 + 6 = 0$

答案正確

由以上的例子，S_n可能先上升，然後下降。

30. 已知$a_5 = 2, a_7 = 1,$ 求a_1及d

$a_i - a_j = (i-j)d$

$\therefore a_5 - a_7 = (5-7)d$

$2 - 1 = -2d$

$d = -\frac{1}{2}$

$a_5 = a_1 + 4d = a_1 + 4\left(-\frac{1}{2}\right) = a_1 - 2 = 2$

$\therefore a_1 = 4$

驗證

$a_7 = a_1 + 6d = 4 + 6\left(-\frac{1}{2}\right) = 4 - 3 = 1$

答案正確

31. 已知$a_6 = 2, a_8 = 4$, 求a_1及d

$a_i - a_j = (i - j)d$

$\therefore a_6 - a_8 = (6 - 8)d$

$2 - 4 = -2d$

$d = 1$

$a_6 = a_1 + 5d = a_1 + 5 = 2$

$\therefore a_1 = -3$

驗證

$a_8 = a_1 + 7d = -3 + 7 = 4$

答案正確

32. 已知$S_6 = -33, S_9 = -90$, 求a_1及d

$S_n = \frac{n}{2}(2a_1 + (n - 1)d)$

$S_6 = \frac{6}{2}(2a_1 + 5d)$

$-33 = 3(2a_1 + 5d)$

$(2a_1 + 5d) = -11 \ldots\ldots (1)$

$S_9 = \frac{9}{2}(2a_1 + 8d)$

$-90 = \frac{9}{2}(2a_1 + 8d)$

$(2a_1 + 8d) = -20 \ldots\ldots (2)$

$(1) - (2)$

$-3d = -11 - (-20) = 9$

$d = -3$　　代入(1)

$2a_1 + 5(-3) = -11$

$2a_1 = -11 + 15 = 4$

$a_1 = 2$

驗證

$S_9 = \frac{9}{2}(2a_1 + 8d)$

$= \frac{9}{2}\big(2(2) + 8(-3)\big)$

$= -90$

答案正確

33. 已知$S_4=-16, S_7=-49$, 求a_1及d

$$S_n=\frac{n}{2}(2a_1+(n-1)d)$$

$$S_4=\frac{4}{2}(2a_1+3d)$$

$-16=2(2a_1+3d)$

$(2a_1+3d)=-8 \ldots\ldots(1)$

$$S_7=\frac{7}{2}(2a_1+6d)$$

$$-49=\frac{7}{2}(2a_1+6d)$$

$(2a_1+6d)=-14 \ldots\ldots(2)$

$(1)-(2)$

$-3d=-8-(-14)=6$

$d=-2$　　代入(1)

$2a_1+3(-2)=-8$

$2a_1=-8+6=-2$

$a_1=-1$

驗證

$$S_7=\frac{7}{2}(2a_1+6d)$$

$$=\frac{7}{2}\big(2(-1)+6(-2)\big)$$

$=-49$

答案正確

$a_1, a_2, \cdots\cdots, a_m$為一等差級數，差是$d_1$

$b_1, b_2, \cdots\cdots, b_n$也是一等差級數，差是$d_2$

$a_1=b_1$

求此二等差級數的共同項目

令C等於d_1和d_2的最小公倍數，

則$C_1=a_1, C_2=C_1+C, C_3=C_1+2C, \cdots\cdots, C_n=C_1+(n-1)C$為二等差級數的共同項目

34. $a_1 = 1, d_1 = 2$
$b_1 = 1, d_2 = 3$

此二等差級數為
1,3,5,7,9,11,13,15,17,19,21,23,25
1,4,7,10,13,16,19,22,25,28,31,34,37

$d_1 = 2$和$d_2 = 3$的最小公倍數是6，所以共同項目是
$1 + 6 = 7, 7 + 6 = 13, 13 + 6 = 19, 19 + 6 = 25$

各位可以看出以上的數字都出現在兩個等差級數之中，而且這些共同項目也是一個等差級數。

35. 一個特別的等差級數
$\mathrm{d} = a_1$
等差級數是$a_1, 2a_1, 3a_1, \ldots\ldots, na_1$

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

$$= \frac{n}{2}\left(2a_1 + a_1(n-1)\right)$$

$$= \frac{n}{2}(2a_1 + na_1 - a_1)$$

$$= \frac{n}{2}(n+1)a_1$$

36. 根據上題，我們可以考慮以下這個特別的等差級數

$$a_1 = \frac{1}{m}, d = \frac{1}{m}, n = m - 1$$

此等差級數是

$$\frac{1}{m}, \frac{2}{m}, \dots\dots, \frac{m-1}{m}$$

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

$$= \frac{m-1}{2}\left(2\left(\frac{1}{m}\right) + (m-1-1)\left(\frac{1}{m}\right)\right)$$

$$= \frac{m-1}{2}\left(\left(\frac{2}{m}\right) + (m-2)\left(\frac{1}{m}\right)\right)$$

$$= \frac{m-1}{2}(1)$$

$$= \frac{m-1}{2}$$

驗證$m = 5$

$$\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} = \frac{1+2+3+4}{5} = 2$$

$$\frac{m-1}{2} = \frac{5-1}{2} = 2$$

故答案正確

此題的答案可以利用第 35 題的答案

$$a_1 = \frac{1}{m}, d = \frac{1}{m}, n = m - 1$$

$$\therefore S_n = \frac{n}{2}(n+1)a_1$$

$$= \frac{m-1(m-1+1)\left(\frac{1}{m}\right)}{2}$$

$$= \frac{(m-1)m}{2}\left(\frac{1}{m}\right)$$

$$= \frac{m-1}{2}$$

37. $a_1 = \frac{1}{m^2}, d = \frac{1}{m^2}, n = m^2 - 1$

$$\therefore S_n = \frac{n}{2}(n+1)a_1$$

$$= \frac{(m^2-1)(m^2-1+1)}{2}\left(\frac{1}{m^2}\right)$$

$$= \frac{m^2-1}{2}$$